

# A NOTE ON THE COLORFUL FRACTIONAL HELLY THEOREM

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**ABSTRACT.** Helly's theorem is a classical result concerning the intersection patterns of convex sets in  $\mathbb{R}^d$ . Two important generalizations are the colorful version and the fractional version. Recently, Bárány et al. combined the two, obtaining a colorful fractional Helly theorem. In this paper, we give an improved version of their result.

## 1. INTRODUCTION

Helly's theorem is one of the most well-known and fundamental results in combinatorial geometry, which has various generalizations and applications. It was first proved by Helly [10] in 1913, but his proof was not published until 1923, after alternative proofs by Radon [15] and König [13]. Recall that a family is *intersecting* if the intersection of all members is non-empty. The following is the original version of Helly's theorem.

**Theorem 1.1** (Helly's theorem, Helly [10]). *Let  $\mathcal{F}$  be a finite family of convex sets in  $\mathbb{R}^d$  with  $|\mathcal{F}| \geq d + 1$ . Suppose every  $(d + 1)$ -tuple of  $\mathcal{F}$  is intersecting. Then the whole family  $\mathcal{F}$  is intersecting.*

The following variant of Helly's theorem was found by Lovász, whose proof appeared first in Bárány's paper [4]. Note that the original Helly's theorem is obtained by setting  $\mathcal{F}_1 = \mathcal{F}_2 = \dots = \mathcal{F}_{d+1}$ .

**Theorem 1.2** (Colorful Helly theorem, Lovász [4]). *Let  $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{d+1}$  be finite, non-empty families (color classes) of convex sets in  $\mathbb{R}^d$  such that every colorful  $(d + 1)$ -tuple is intersecting. Then for some  $1 \leq i \leq d + 1$ , the whole family  $\mathcal{F}_i$  is intersecting.*

One way to generalize Helly's theorem is by weakening the assumption: not necessarily all but only a positive fraction of  $(d + 1)$ -tuples are intersecting. The following theorem shows how the conclusion changes.

**Theorem 1.3** (Fractional Helly theorem, Katchalski and Liu [12]). *For every  $\alpha \in (0, 1]$ , there exists  $\beta = \beta(\alpha, d) \in (0, 1]$  such that the following holds: Let  $\mathcal{F}$  be a finite family of convex sets in  $\mathbb{R}^d$  with  $|\mathcal{F}| \geq d + 1$ . If at least  $\alpha \binom{|\mathcal{F}|}{d+1}$  of the  $(d + 1)$ -tuples in  $\mathcal{F}$  are intersecting, then  $\mathcal{F}$  contains an intersecting subfamily of size at least  $\beta|\mathcal{F}|$ .*

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The fractional variant of Helly's theorem first appeared as a conjecture on interval graphs, i.e. intersection graphs of families of intervals on  $\mathbb{R}$ . Abbott and Katchalski [1] proved that  $\beta = 1 - \sqrt{1 - \alpha}$  is optimal for every family whose intersection graph is a chordal graph. Note that, by the result of Gavril [8], interval graphs are chordal graphs.

The fractional Helly theorem for arbitrary dimensions was proved by Katchalski and Liu [12]. Their proof gives a lower bound  $\beta \geq \alpha/(d + 1)$ . However, it seems natural that  $\beta$  tends to 1 as  $\alpha$  tends to 1, since the original Helly's theorem implies that  $\beta = 1$  when  $\alpha = 1$ . Later, the quantitatively sharp value  $\beta(\alpha, d) = 1 - (1 - \alpha)^{1/(d+1)}$  was found by Kalai [11] and Eckhoff [6]. The result follows from the upper bound theorem for families of convex sets.

The  $(p, q)$ -theorem, another important generalization of Helly's theorem, deals with a weaker version of the assumption, the so-called  $(p, q)$ -condition: for every  $p$  members in a given family, there are some  $q$  members of the family that are intersecting. For instance, the  $(d + 1, d + 1)$ -condition in  $\mathbb{R}^d$  is the hypothesis of Helly's theorem. The  $(p, q)$ -theorem was proved by Alon and Kleitman [2], settling a conjecture by Hadwiger and Debrunner [9].

The proof of the  $(p, q)$ -theorem is quite long and involved, using various techniques. However, one of the most crucial ingredients is the fractional Helly theorem. See the survey paper by Eckhoff [7] and the textbook by Matousek [14] for an overview and further knowledge of this field.

Recently, Bárány et al. [5] established the colorful and fractional versions of the  $(p, q)$ -theorem. A key ingredient in their proof was a colorful variant of the fractional Helly theorem.

**Theorem 1.4** (Bárány-Fodor-Montejano-Oliveros-Pór [5]). *Let  $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{d+1}$  be finite, non-empty families (color classes) of convex sets in  $\mathbb{R}^d$ , and assume that  $\alpha \in (0, 1]$ . If at least  $\alpha|\mathcal{F}_1| \cdots |\mathcal{F}_{d+1}|$  of the colorful  $(d + 1)$ -tuples are intersecting, then some  $\mathcal{F}_i$  contains an intersecting subfamily of size  $\frac{\alpha}{d+1}|\mathcal{F}_i|$ .*

Note that for  $\alpha = 1$  we recover the hypothesis of the colorful Helly theorem, and we should therefore expect the value  $\beta = 1$  (rather than  $\beta = \frac{1}{d+1}$ ). Bárány et al. therefore proposed the problem of showing that the function  $\beta$  in Theorem 1.4 tends to 1 as  $\alpha$  tends to 1.

In this paper, we solve the problem of Bárány et al.

**Theorem 1.5.** *For every  $\alpha \in (0, 1]$ , there exists  $\beta = \beta(\alpha, d) \in (0, 1]$  tending to 1 as  $\alpha$  tends to 1 such that the following holds: Let  $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{d+1}$  be finite, non-empty families (color classes) of convex sets in  $\mathbb{R}^d$ . If at least  $\alpha|\mathcal{F}_1| \cdots |\mathcal{F}_{d+1}|$  of the colorful  $(d + 1)$ -tuples are intersecting, then for some  $1 \leq i \leq d + 1$ ,  $\mathcal{F}_i$  contains an intersecting subfamily of size  $\beta|\mathcal{F}_i|$ .*

See the survey paper by Amenta, Loera, and Soberón [3] for an overview of recent results and open problems related to Helly's theorem.

## 2. PROOF OF THEOREM 1.5

**2.1. The Matching number of hypergraphs.** Let  $\mathcal{H}$  be an  $r$ -uniform hypergraph on a vertex set  $X$ . A subset  $S \subseteq X$  is said to be an *independent set* in  $\mathcal{H}$  if the induced sub-hypergraph  $\mathcal{H}[S]$  contains no hyperedge. The *independence number*  $\alpha(\mathcal{H})$  of  $\mathcal{H}$  is the cardinality of a maximum independent set in  $\mathcal{H}$ . A *matching* of  $\mathcal{H}$  is a set of pairwise disjoint edges in  $\mathcal{H}$ . The *matching number*  $\nu(\mathcal{H})$  of  $\mathcal{H}$  is the cardinality of a maximum matching in  $\mathcal{H}$ . For our result, we need the following observation.

**Observation 2.1.** Let  $\mathcal{H} = (X, E)$  be an  $r$ -uniform hypergraph with  $|X| = n$ . Suppose  $\alpha(\mathcal{H}) < cn$  for some  $c \in (0, 1]$ . Let  $M$  be a maximum matching in  $\mathcal{H}$ . Note that  $X \setminus M$  is an independent set in  $\mathcal{H}$ . If not, assume that there is an edge  $e$  contained in  $X \setminus M$ . Then  $M \cup \{e\}$  is a matching in  $\mathcal{H}$ , which is a contradiction to the maximality of  $M$ . Thus  $|X \setminus M| = n - r\nu(\mathcal{H}) \leq \alpha(\mathcal{H}) < cn$ , so  $\nu(\mathcal{H}) > \frac{n-cn}{r}$ .

**2.2. Proof of Theorem 1.5.** Theorem 1.6 is implied by the following more explicit result.

**Theorem 2.2.** *For every  $\alpha \in (0, 1]$ , the following holds: Let  $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{d+1}$  be finite families (color classes) of convex sets in  $\mathbb{R}^d$ . If at least  $\alpha|\mathcal{F}_1| \cdots |\mathcal{F}_{d+1}|$  of the colorful  $(d+1)$ -tuples are intersecting, then for some  $1 \leq i \leq d+1$ ,  $\mathcal{F}_i$  contains an intersecting subfamily of size at least*

$$\max \left\{ \frac{\alpha}{d+1}, 1 - (d+1)(1-\alpha)^{\frac{1}{d+1}} \right\} |\mathcal{F}_i|.$$

The following is a key lemma of the proof of Theorem 2.2.

**Lemma 2.3.** *Choose any  $(d+1)$  members from each color class, say  $\mathcal{F}'_1, \dots, \mathcal{F}'_{d+1}$ . If each of  $\mathcal{F}'_i$  is not intersecting, then at least one of colorful  $(d+1)$ -tuple is not intersecting.*

*Proof.* This follows directly from the colorful Helly theorem.  $\square$

*Proof of Theorem 2.2.* It is sufficient to show that for every  $\alpha \in [1 - \frac{1}{(d+1)^{(d+1)}}, 1]$ , if at least  $\alpha|\mathcal{F}_1| \cdots |\mathcal{F}_{d+1}|$  of the colorful  $(d+1)$ -tuples are intersecting, then some  $\mathcal{F}_i$  contains an intersecting subfamily of size at least  $(1 - (d+1)(1-\alpha)^{\frac{1}{d+1}})|\mathcal{F}_i|$ .

Let  $\mathcal{F}$  be the disjoint union of  $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{d+1}$ . For each  $1 \leq i \leq d+1$ , denote  $n_i = |\mathcal{F}_i|$  and define a  $(d+1)$ -uniform hypergraph  $H_i := (\mathcal{F}_i, E_i)$  whose vertices are the members in  $\mathcal{F}_i$  and hyperedges are non-intersecting  $(d+1)$ -tuples in  $\mathcal{F}_i$ . Let  $\nu_j = \nu(H_j)$  for each  $1 \leq j \leq d+1$ .

Also define a  $(d+1)$ -uniform hypergraph  $H := (\mathcal{F}, E)$  whose vertices are the members in  $\mathcal{F}$  and hyperedges are intersecting colorful  $(d+1)$ -tuples in  $\mathcal{F}$ .

Given  $\alpha \in [1 - \frac{1}{(d+1)^{d+1}}, 1]$ , let  $\gamma = \gamma(\alpha, d) = 1 - (d+1)(1-\alpha)^{\frac{1}{d+1}}$ . For contraction, assume that in each family  $\mathcal{F}_j$ , every subfamily of size at least  $\gamma n_j$  has an empty intersection.

By Lemma 2.3, we have  $\alpha n_1 \cdots n_{d+1} \leq |E| \leq n_1 \cdots n_d + 1 - v_1 \cdots v_{d+1}$ . Recall that by Observation 2.1,  $v_j > \frac{n_j - \gamma n_j}{d+1} = \left(\frac{1-\gamma}{d+1}\right) n_j$  for each  $1 \leq j \leq d+1$ . Then we obtain

$$\begin{aligned} \alpha n_1 \cdots n_{d+1} &\leq n_1 \cdots n_{d+1} - v_1 \cdots v_{d+1} \\ &< n_1 \cdots n_{d+1} - \left(\frac{1-\gamma}{d+1}\right)^{d+1} n_1 \cdots n_{d+1} \\ &= \left(1 - \left(\frac{1-\gamma}{d+1}\right)^{d+1}\right) n_1 \cdots n_{d+1}, \end{aligned}$$

hence  $\alpha < 1 - \left(\frac{1-\gamma}{d+1}\right)^{d+1} = \alpha$ , which is a contradiction.

Thus, there should exist  $1 \leq i \leq d+1$  such that  $\mathcal{F}_i$  contains an intersecting subfamily of size  $(1 - (d+1)(1-\alpha)^{\frac{1}{d+1}})n_i$ .  $\square$

### 3. THE UPPER BOUND

First recall that in the fractional Helly theorem, the upper bound is given by

$$\beta = \beta(\alpha, d) \leq (1 - (1 - \alpha)^{\frac{1}{d+1}}).$$

This can be seen by the following well-known construction, which also shows the exactness of upper bound theorem for convex sets [6][11].

**Example 3.1.** Let  $\mathcal{F}$  consist of  $\lfloor \beta n \rfloor - (d+1)$  copies of  $\mathbb{R}^d$  and  $n - \lfloor \beta n \rfloor + (d+1)$  hyperplanes in general position. Denote by  $f_d(\mathcal{F})$  the number of intersecting  $(d+1)$ -tuples in  $\mathcal{F}$ . Note that

$$\begin{aligned} \alpha \binom{n}{d+1} = f_d(\mathcal{F}) &= \binom{n}{d+1} - \binom{n - (\lfloor \beta n \rfloor - (d+1))}{d+1} \\ &< \binom{n}{d+1} - \binom{n - \lfloor \beta n \rfloor}{d+1} \\ &\leq \binom{n}{d+1} - (1 - \beta)^{d+1} \binom{n}{d+1}. \end{aligned}$$

The colorful version of this example gives an upper bound for the colorful fractional Helly theorem.

**Theorem 3.2.** *For every  $\alpha \in (0, 1]$ , there exist finite families (color classes)  $\mathcal{F}_1, \dots, \mathcal{F}_{d+1}$  of convex sets in  $\mathbb{R}^d$  such that the following holds.  $\alpha |\mathcal{F}_1| \cdots |\mathcal{F}_{d+1}|$  of the colorful  $(d+1)$ -tuples are intersecting, but in each color class  $\mathcal{F}_i$ , the maximum cardinality of an intersecting subfamily is at most  $(1 - (1 - \alpha)^{\frac{1}{d+1}}) |\mathcal{F}_i|$ .*

*Proof.* It follows from the following construction. Let  $\mathcal{F}_i$  consist of  $\lfloor \beta n \rfloor - d$  copies of  $\mathbb{R}^d$  and  $n - \lfloor \beta n \rfloor + d$  hyperplanes in general position. Moreover, let all hyperplanes in  $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_{d+1}$  be in general position. Note that each of  $\mathcal{F}_i$  has an intersecting subfamily of size at most  $\beta n$ . The number of colorful  $(d + 1)$ -tuples is given by

$$\alpha n^{d+1} = n^{d+1} - (n - \lfloor \beta n \rfloor + d)^{d+1} < n^{d+1} - (1 - \beta)^{d+1} n^{d+1}.$$

As  $n$  tends to infinity, one may have that  $\alpha = 1 - (1 - \beta)^{d+1}$ , i.e.  $\beta = 1 - (1 - \alpha)^{\frac{1}{d+1}}$ .  $\square$

#### 4. REMARKS

In this note, we found upper and lower bounds on the function  $\beta(\alpha, d)$  in the colorful fractional Helly theorem, however, there remains a large gap between them. It would be interesting to determine the exact value of  $\beta(\alpha, d)$ .

**Problem 4.1.** What is the exact value of  $\beta = \beta(\alpha, d)$  in Theorem 1.5?

It is easy to see that  $\beta(\alpha, 1) = 1 - \sqrt{1 - \alpha}$  is the optimal bound for  $d = 1$ . We conjecture that  $\beta(\alpha, d) = 1 - (1 - \alpha)^{\frac{1}{d+1}}$  is the optimal bound for  $d > 1$ .

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